Some Problems with Unexpected Tiling Solutions

Grant Mullins

Virginia Tech Math Club

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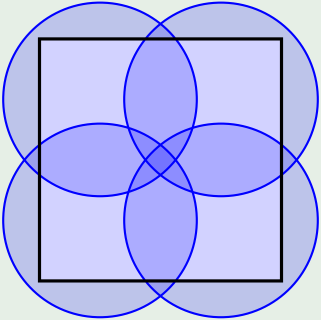
Introduction to Tilings

What is a Covering?

A **covering** of a region S with tiles from a set $\mathcal{T}$ is a collection $\mathcal{C}$ of regions, each congruent to a tile in $\mathcal{T}$, such that $S \subseteq \cup_{C \in \mathcal{C}} C$.

An Example Covering

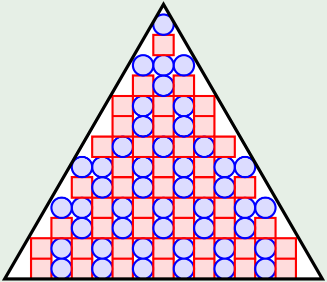


Introduction to Tilings

What is a Packing?

A **packing** of a region S with tiles from a set $\mathcal{T}$ is a collection $\mathcal{C}$ of regions, each a subset of S and congruent to a tile in $\mathcal{T}$, such that $\text{Int}(C_1) \cap \text{Int}(C_2) = \emptyset$ for all distinct $C_1, C_2 \in \mathcal{C}$.

An Example Packing

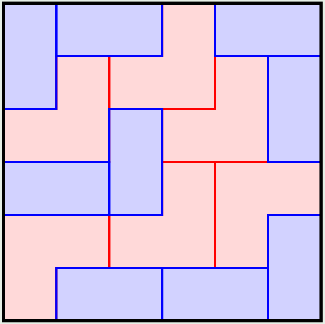


Introduction to Tilings

What is a Tiling?

A **tiling** of a region S with tiles from a set $\mathcal{T}$ is a covering and a packing of S with tiles from $\mathcal{T}$.

An Example Tiling



A Game on Grids

Problem 1

Alice and Bob play a game starting at the corner square of an $n \times n$ grid. A legal move is to move to an adjacent grid square that has not been visited before. Alice goes first, and the players alternate moves. The first player to have no legal move loses. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Small Values of n

When $n = 1$, Bob wins since Alice cannot move. When $n = 2$, we can see that Alice always wins. For $n \geq 3$, the possibilities get harder to keep track of. We can look at an example game when $n = 3$.

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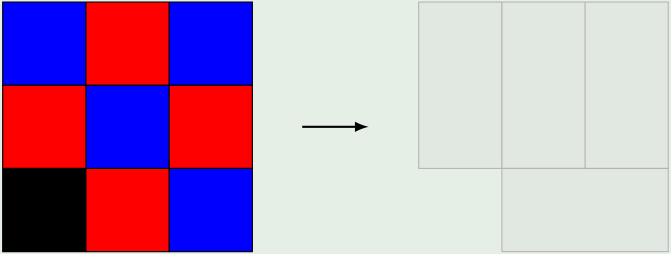
An example $n = 3$ Game



A Game on Grids

Problem 1

Alice and Bob play a game starting at the corner square of an $n \times n$ grid. A legal move is to move to an adjacent grid square that has not been visited before. Alice goes first, and the players alternate moves. The first player to have no legal move loses. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

The Connection To Tiling

Claim: If the portion of the $n \times n$ grid that has not yet been visited can be tiled with 1×2 tiles, then the player that moves second from that position has a winning strategy.

Proof: The player that moves second can force each of the first player's moves to enter a 1×2 tile that has not yet been visited. They can do this by moving to the other grid square within each 1×2 tile. This means the second player always has a legal move, so they have the winning strategy.

A Game on Grids

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Alice and Bob play a game starting at the corner square of an $n \times n$ grid. A legal move is to move to an adjacent grid square that has not been visited before. Alice goes first, and the players alternate moves. The first player to have no legal move loses. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Tiling When n is Odd

Claim: If n is odd, then an $n \times n$ grid with a corner removed can be tiled with 1×2 tiles.

Proof: Assume the removed grid square is the bottom left square. Tile the leftmost column above the removed square with $(n-1)/2$ tiles, and tile the bottom row to the right of the removed square with $(n-1)/2$ tiles. Thus, the remaining $(n-1) \times (n-1)$ grid can be tiled with 1×2 tiles since $n-1$ is even.

A Game on Grids

Problem 1

Alice and Bob play a game starting at the corner square of an $n \times n$ grid. A legal move is to move to an adjacent grid square that has not been visited before. Alice goes first, and the players alternate moves. The first player to have no legal move loses. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Tiling When n is Even

Claim: If n is even, then an $n \times n$ grid with a corner and adjacent grid square removed can be tiled with 1×2 tiles.

Proof: We can tile the entire $n \times n$ grid with tiles since n is even. By diagonal reflection, we can assume the removed squares lie in the same 1×2 tile. Thus, the remaining grid squares remain tiled.

A Game on Grids

Problem 1

Alice and Bob play a game starting at the corner square of an $n \times n$ grid. A legal move is to move to an adjacent grid square that has not been visited before. Alice goes first, and the players alternate moves. The first player to have no legal move loses. For each $n \geq 1$, determine which player has a strategy that guarantees winning.

Wrapping Things Up

If n is odd, then we can tile the grid with the starting corner removed with 1×2 tiles. Hence the player that moves second, which is Bob, has a winning strategy.

If n is even, then after Alice makes an initial move, we can tile the remaining grid with 1×2 tiles. Hence the player that moves second from this position, which is Alice, has a winning strategy. ■

A Fibonacci Identity

Problem 2

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all positive integers n.

Exploration

$$F_1^2 + F_0^2 = 1^2 + 0^2 = 1 = F_1$$
$$F_2^2 + F_1^2 = 1^2 + 1^2 = 2 = F_3$$
$$F_3^2 + F_2^2 = 2^2 + 1^2 = 5 = F_5$$
$$F_4^2 + F_3^2 = 3^2 + 2^2 = 13 = F_7$$
$$F_5^2 + F_4^2 = 5^2 + 3^2 = 34 = F_9$$
$$F_6^2 + F_5^2 = 8^2 + 5^2 = 89 = F_{11}$$

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The Connection to Tiling

How many different ways are there to tile a $1 \times n$ grid with 1×1 and 1×2 tiles?

Let T_n denote the number of ways to tile a $1 \times n$ grid as above.

We will compute T_n in two different ways. The first way will connect T_n with the Fibonacci numbers, and the second way will prove our identity.

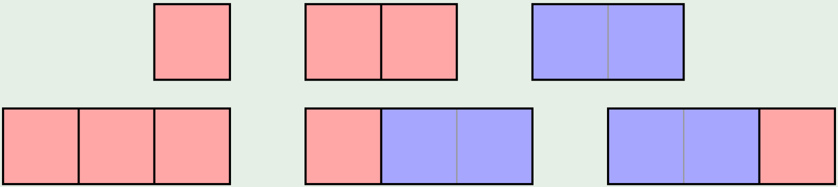
A Fibonacci Identity

Problem 2

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all positive integers n.

Computing T_n

We can compute $T_1 = 1$, $T_2 = 2$, and $T_3 = 3$.



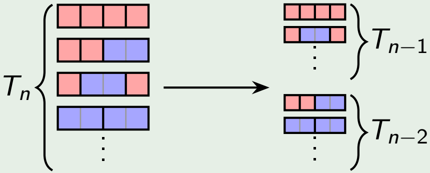
A Fibonacci Identity

Problem 2

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all positive integers n.

Finding a Recursive Relation for T_n

For a fixed $n \geq 4$, consider the set of tilings of a $1 \times n$ grid using 1×1 and 1×2 tiles. By definition, there are T_n tilings in this set. Partition this set into two based on the last tile. Thus, $T_n = T_{n-1} + T_{n-2}$ for all $n \geq 4$.



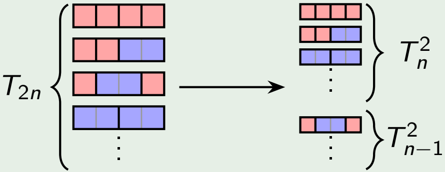
A Fibonacci Identity

Problem 2

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all positive integers n.

Proving the Identity for T_n

For a fixed $n \geq 2$, consider the set of tilings of a $1 \times 2n$ grid using 1×1 and 1×2 tiles. By definition, there are T_{2n} tilings in this set. Partition this set into two based on whether the middle two squares are covered by a single 1×2 tile. Thus, $T_{2n} = T_n^2 + T_{n-1}^2$ for all $n \geq 2$.



A Fibonacci Identity

Problem 2

Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_n^2 + F_{n-1}^2 = F_{2n-1}$ for all positive integers n.

Wrapping Things Up

We now have all the information we need about the sequence T_n:

1. $T_1 = 1$, $T_2 = 2$, and $T_n = T_{n-1} + T_{n-2}$ for all $n \geq 3$.
2. $T_{2n} = T_n^2 + T_{n-1}^2$ for all $n \geq 2$.

The first condition implies that $T_n = F_{n+1}$ for all $n \geq 1$. Plugging this into the second condition yields $F_{2n+1} = F_{n+1}^2 + F_n^2$ for all $n \geq 2$. Shifting the index down by 1 and using our computation for the first two cases yields $F_{2n-1} = F_n^2 + F_{n-1}^2$ for all positive integers n. ∎

Distance Between Points in the Plane

Notation

For a set X of points in the plane, let $\delta(X)$ denote the distance between the closest pair of distinct points in X.

Problem 3

Let S be a set of 100 distinct points in the plane with $\delta(S) = 2$. Prove that there exists a subset $T \subset S$ of 15 distinct points with $\delta(T) > \sqrt{7}$.

Initial Thoughts

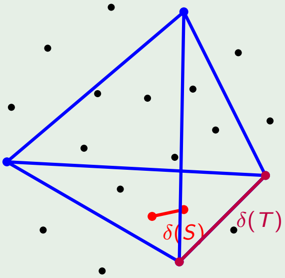
1. Given a set of points, we need to show that we can choose a subset of the points that are more spaced out.
2. We can work out an example to get a better grasp of the problem.

Distance Between Points in the Plane

Problem 3

Let S be a set of 100 distinct points in the plane with $\delta(S) = 2$. Prove that there exists a subset $T \subset S$ of 15 distinct points with $\delta(T) > \sqrt{7}$.

A Small Example



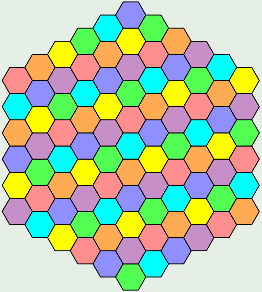
Distance Between Points in the Plane

Problem 3

Let S be a set of 100 distinct points in the plane with $\delta(S) = 2$. Prove that there exists a subset $T \subset S$ of 15 distinct points with $\delta(T) > \sqrt{7}$.

The Connection to Tiling

We can tile the plane with hexagons.



Distance Between Points in the Plane

Problem 3

Let S be a set of 100 distinct points in the plane with $\delta(S) = 2$. Prove that there exists a subset $T \subset S$ of 15 distinct points with $\delta(T) > \sqrt{7}$.

Tiling and Coloring the Plane

Tile the plane with hexagons of diameter $\delta(S) = 2$, and color them with 7 colors as shown previously. By shifting the tiling, we may assume that each point in S lies in the interior of some hexagon.

Each of the 100 points is colored one of 7 colors. By the pigeonhole principle, there exist 15 points that are all the same color. Take T to be the set of these 15 points. We are left to show that $\delta(T) > \sqrt{7}$.

Problem 3

Let S be a set of 100 distinct points in the plane with $\delta(S) = 2$. Prove that there exists a subset $T \subset S$ of 15 distinct points with $\delta(T) > \sqrt{7}$.

Bounding $\delta(T)$

Since each hexagon has diameter $\delta(S) = 2$, no two points share a hexagon. Hence $\delta(T)$ is greater than the closest distance between any two same-colored hexagons. Thus, $\delta(T) > \sqrt{7}$. ■