Putnam Prep

Grant Mullins, Virginia Tech Math Club

Fall 2024

Contents

1 Important Tools

Induction Let $P(x)$ be a mathematical statement. If $P(n_0)$ is true for some $n_0 \in \mathbb{Z}$ and $P(n) \Rightarrow P(n+1)$ for all $n \geq n_0$, then $P(n)$ is true for all $n \geq n_0$.

1. Prove that every integer $n \geq 8$ can be written as a sum of at least two (not necessarily distinct) odd prime numbers.

2. Let $n \geq 2$ be an integer. Prove that
$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}.$$

3. If $n \geq 3$ lines are drawn in the plane so that no two are parallel and no three meet at a single point, then show that at least one of the regions the lines partition the plane into is a triangle.

4. Alice and Bob play a game with a pile of $n \geq 1$ rocks. They take turns taking exactly 1, 3, or 4 rocks from the pile, and the winner is the player to take the last rock. If Alice goes first, for what n does Alice have a winning strategy (a strategy so that, no matter what moves Bob makes, Alice is guaranteed to win)?

5. Let $x_0, x_1, x_2, \ldots$ be an infinite sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for all $n \geq 1$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

6. Prove that a list can be made of all the subsets of a finite set in such a way that (**i**) the empty set is first in the list, (**ii**) each subset occurs exactly once, (**iii**) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

Pigeonhole If $m > n$ objects are distributed into n boxes, then there exists a box with at least 2 objects. More generally, if $m > nk$ objects are distributed into n boxes, then there exists a box with at least $k+1$ objects.

1. Five points are placed within a unit square (square of side length 1). Prove at least one pair of points are within $1/\sqrt{2}$ units of each other.

2. For $n \geq 1$, let S be a subset of $\{1, 2, \ldots, 2n\}$ containing $n+1$ elements.

   (a) Prove there exist distinct $a, b \in S$ such that $\gcd(a, b) = 1$.

   (b) Prove there exist distinct $a, b \in S$ such that $\gcd(a, b) = a$.

3. Suppose 5 points lie on the surface of a sphere. Prove that there exists a closed semi-sphere (half of the sphere including the boundary) which contains at least 4 of the points.

4. Let $S = \{P_1, \dots, P_9\}$ be any set of 9 distinct lattice points (points with integer coordinates) in three dimensional Euclidean space. Show that there exists a lattice point $P \notin S$ that lies on a line segment with points $P_i, P_j \in S$ as endpoints.

5. A pie-lover decides to eat exactly 45 pies over a span of 30 days. They pace themselves by eating at least 1 pie on each of the 30 days. Prove there exists some period of consecutive days where the pie-lover eats exactly 14 pies.

Invariants An invariant is a characteristic or quantity that remains unchanged after some choice or operation. This often involves the parity or modulus of a quantity, and is useful when showing something is not possible.

1. Find the number of ways to choose $c_i \in \{-1, 1\}$ for $1 \le i \le 1234$ such that $1c_1 + 2c_2 + \cdots + 1234c_{1234} = 0$.

2. Consider an 8×8 checkerboard with two opposite corner squares removed. Is it possible to tile this checkerboard with non-overlapping 2×1 dominoes?

3. Start with an ordered triple of numbers. An operation consists of choosing any two of the numbers, say a and b, and replacing one with $(a+b)/\sqrt{2}$ and the other with $(a-b)/\sqrt{2}$. Is it possible to transform $(1, \sqrt{2}, 1+\sqrt{2})$ into $(2, \sqrt{2}, 1/\sqrt{2})$ using finitely many operations?

4. The integer $10^{2024} - 1$ is written on a chalkboard. On each turn, one integer x on the chalkboard is chosen and two integers y and z are chosen such that $yz = x$. Then, x is erased, and one integer from $\{y-2, y+2\}$ and another from $\{z-2, z+2\}$ are written on the chalkboard. Is it possible that, after a finite number of turns, the only integers written on the chalkboard are 9's?

5. Start with a sequence $a_1, a_2, \dots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

Induction hints:

1. How can you construct the sum for 11 using the sum for 8?

2. Write the desired left hand side for the $n+1$ case. What terms overlap with the n case?

3. If we add a line to a plane of n lines that contain a triangle region, it can either intersect the triangle or not.

4. Work out the cases for small n. What does it mean for some n to not have a winning strategy?

5. (1993 A2) Define the sequence $a_n = (x_{n+1} + x_{n-1})/x_n$ for $n \geq 1$. Prove a_n is a constant sequence.

6. (1968 A3) Let $S = \{s_1, \ldots, s_{n+1}\}$ be a set with $n+1$ elements. How many subsets contain s_{n+1}?

Pigeonhole hints:

1. (1954 A2) Partition the square into four quadrants.

2. (a) Partition the set into the n sets $\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}$.
   (b) (1958 B2) Let $o(m)$ denote the largest odd divisor of m. Show there exist $x, y \in S$ such that $o(x) = o(y)$.

3. (2002 A2) Can you always choose a semi-sphere so that two points lie on its boundary?

4. (1971 A1) When does the line segment between lattice points $(0,0,0)$ and (x,y,z) contain a lattice point?

5. (*Problem-Solving Strategies*) Let S_n denote the sum of the number of pies eaten across days $1, 2, \ldots, n$, inclusive. We must show $S_i = S_j + 14$ for some $1 \leq i < j \leq 30$. What values can S_n and $S_n + 14$ be for $1 \leq n \leq 30$?

Invariants hints:

1. Consider the parity of the sum.

2. What color squares does a 2×1 domino cover?

3. After an operation, what must the sum of the squares of the entries be?

4. $10^{2024} - 1$ is three more than a multiple of 4. What factors can such numbers have?

5. (2008 A3) What is the product of the integers after an operation? What about their sum?

*Italicized problem sources are slightly modified or served as inspiration. Problems with no source are original, classics, or problems I have forgotten the source of.

2 Polynomials

Roots and Factorization Let $p(x)$ be an n-degree polynomial with real coefficients and (not necessarily real) roots $r_1, \dots, r_k$. Then there exists a unique $(n-k)$-degree polynomial $q(x)$ such that

$$p(x) = q(x)(x-r_1)\cdots(x-r_k).$$

Moreover, if we know all n roots $r_1, \dots, r_n$, then for some $a \in \mathbb{R}$,

$$p(x) = a(x-r_1)\cdots(x-r_n).$$

1. Let p be a real polynomial with degree n. Prove that p has at most n complex roots.
2. Let $p(x)$ be a monic (leading coefficient of 1) real polynomial of degree four. If $p(1) = 2$, $p(2) = 4$, and $p(3) = 6$, calculate $p(6) + p(-2)$.
3. Let $p(x)$ be a polynomial with real coefficients such that r is a root of p with multiplicity exactly $n \geq 2$. Prove that r is a root of $p'(x)$ with multiplicity exactly $n-1$.
4. Determine all real polynomials $P(x)$ such that $P(0) = 0$ and $P(x^2+1) = (P(x))^2 + 1$ for all $x \in \mathbb{R}$.
5. Let $P(x)$ be a real polynomial of degree n with roots $r_1, r_2, \dots, r_n$. Prove
$$\frac{1}{x-r_1} + \frac{1}{x-r_2} + \cdots + \frac{1}{x-r_n} = \frac{P'(x)}{P(x)}$$
for all x with $P(x) \neq 0$.

Vieta's Formulas Let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients and roots $r_1, \dots, r_n$. Then

$$\begin{aligned}
-a_{n-1} &= r_1 + \cdots + r_n \\
a_{n-2} &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n \\
-a_{n-3} &= r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + r_{n-2} r_{n-1} r_n \\
&\vdots \\
(-1)^n a_0 &= r_1 \cdots r_n
\end{aligned}$$

where each sum runs through all possible k-tuples of roots, $1 \leq k \leq n$. This follows directly by writing $p(x) = (x-r_1)\cdots(x-r_n)$ and expanding.

1. Suppose $p(x) = x^3 + 2x^2 + 3x + 4$ has roots r_1, r_2, and r_3, and $q(x) = x^3 + ax^2 + bx + c$ has roots $r_1 + r_2$, $r_2 + r_3$, and $r_3 + r_1$. Calculate $q(1)$.
2. Let $p(x) = a_{99}x^{99} + \cdots + a_1 x + a_0$ be a polynomial such that $p(1) = 1^{100}$, $p(2) = 2^{100}, \ldots, p(100) = 100^{100}$. Find a_{99}.
3. Consider the lines that meet the graph $y = 2x^4 + 7x^3 + 3x - 5$ in four distinct points $P_i = (x_i, y_i)$, $i = 1, 2, 3, 4$. Prove that
$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$
is independent of the line, and compute its value.
4. Three distinct real numbers a, b, c satisfy
$$a - \frac{1}{a} + \frac{1}{a^2} = b - \frac{1}{b} + \frac{1}{b^2} = c - \frac{1}{c} + \frac{1}{c^2}.$$
Prove that $1/a + 1/b + 1/c = 1$.
5. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.
6. Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \le |k| \le n$. Find all other real numbers x for which $p(1/x) = x^2$.

Rational Root Theorem

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in \mathbb{Z}$ and $a_n, a_0 \neq 0$. If r/s is a rational root of $p(x)$, where $\gcd(r, s) = 1$, then $r \mid a_0$ and $s \mid a_n$.

This theorem allows you to search for all possible rational roots of an integer polynomial by trial and error and polynomial division.

1. Find all real roots of $3x^3 - 5x^2 + 5x - 2$.
2. Solve the system $3(x + y + z) = 12(xy + yz + zx) = -2xyz = 12$.
3. Solve the system $\{x\} = \{x^2\} = \{x^3\}$, where $\{y\}$ denotes the fractional part of y (y minus the greatest integer not greater than y).
4. Find a necessary and sufficient condition on the positive integer n so that the equation
$$x^n + (2 + x)^n + (2 - x)^n = 0$$
has a rational root.

Roots and Factorization hints:

1. Prove by contradiction.
2. What are the roots of $q(x) := p(x) - 2x$?
3. Write $p(x) := q(x)(x-r)^n$ and use product rule.
4. (1971 A2) Show that $Q(x) := P(x) - x$ has infinitely many roots.
5. Write $P(x) = a(x-r_1)\cdots(x-r_n)$ and take the logarithm.

Vieta's Formulas hints:

1. Express a, b, c using Vieta's, then use information from $p(x)$.
2. Let $q(x) := p(x) - x^{100}$. What are the roots of q?
3. (1977 A1) Let the line be $y = ax + b$. What does it mean for two functions to intersect?
4. Let the three quantities be equal to k. a, b, c must be three distinct roots of what equation?
5. (1984 USAMO) Let a, b, c, d be the roots with $ab = -32$. Calculate the values of cd, $a+b$, and $c+d$ to determine k.
6. (2023 A2) Write out $q(x) := p(1/x) - x^2$. What are the roots of q?

Rational Root Theorem hints:

1. Rational roots p/q must satisfy $p \mid 2$ and $q \mid 3$. Check these for roots then factor.
2. Construct a polynomial $p(t)$ with roots x, y, z and find its roots by rational root theorem.
3. What must $x^2 - x$ and $x^3 - x^2$ be? Prove x is rational and a root of a monic polynomial.
4. (1955 A6) Show no integer roots exist for even n, then for no odd $n \geq 3$ by proving x must be an integer.

*Problems with no source are original, classics, or problems I have forgotten the source of.

3 Sequences and Series

Monotone Convergence Theorem If a sequence is eventually monotone (either always increasing or always decreasing) and bounded, it must converge. Note that if a sequence (a_n) converges to L, then $\lim_{n\to\infty} a_{n+k} = L$ for any $k \in \mathbb{Z}$.

1. Let $a_0 = 2$ and $a_{n+1} = (a_n + 5)/3$ for $n \geq 1$. Prove that (a_n) converges and find its limit.
2. Prove that $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ converges and determine its value.
3. Prove that $1 + 1/(2!) + 1/(3!) + \cdots$ converges (without the use of e).
4. Show that
$$\sqrt[8]{2207 - \cfrac{1}{2207 - \frac{1}{2207 - \dots}}}$$
converges and find its value. Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.
5. If (a_n) is a sequence of numbers such that for $n \geq 1$
$$(2 - a_n)a_{n+1} = 1,$$
prove that $(a_n) \to 1$ as $n \to \infty$.
6. Let $a_0 \in \mathbb{R}$ and $a_{n+1} = \cos(a_n)$ for $n \geq 1$. Prove that (a_n) converges and its value is independent from a_0.

Telescoping Let (a_n) be a sequence with limit 0. Then the series of the form $(a_0 - a_1) + (a_1 - a_2) + \cdots$ converges to a_0.

Similarly, if a_0/a_n has limit L, then the product of the form $(a_0/a_1)\cdot(a_1/a_2)\cdots$ converges to L.

1. Evaluate
$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$
2. Evaluate
$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots.$$

3. Let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ be the n^{th} harmonic number. Evaluate
$$\sum_{n=1}^{\infty} \frac{1}{(n+1) H_n H_{n+1}}.$$

4. Let (F_n) be the Fibonacci sequence, so $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate
$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1} F_{n+1}}.$$

5. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,
$$x_{n+1} = \ln(e^{x_n} - x_n)$$
where ln denotes the natural logarithm. Show that the infinite series $x_0 + x_1 + x_2 + \cdots$ converges and find its sum.

6. Evaluate the infinite product:
$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

7. Evaluate
$$\sum_{n=0}^{\infty} \frac{2^n}{5^{2^n} + 1}.$$

8. Express
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2 n + m n^2 + 2mn}$$
as a rational number.

Monotone Convergence Theorem hints:

1. Prove $0 < a_n < 5/2$ and $a_{n+1} - a_n > 0$ using induction.
2. Let $a_0 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$. Prove $\{a_n\}$ is increasing and bounded above by 2.
3. Find a larger sum that is easy to calculate.
4. (1995 B4) Show $a_0 = 2207$, $a_{n+1} = 2207 - 1/a_n$ is decreasing and bounded below, and thus converges to L. Set $x^8 = L$ and solve for x.
5. (1947 A1) Show there exists an $m \geq 1$ such that $a_m < 1$. Afterwards the sequence is increasing and bounded above.
6. (1952 B7) Bound a_2. Show $a_2, a_4, a_6, \ldots$ and $a_3, a_5, a_7, \ldots$ each converge to the same value.

Telescoping hints:

1. Rationalize the denominators.
2. Partial fraction decomposition.
3. Guess how the sum should telescope.
4. Rewrite $F_n/(F_{n-1}F_nF_{n+1})$.
5. (2016 B1) Show x_n is decreasing and bounded below. Exponentiate the recurrence relation.
6. (1977 B1) Factor and split the partial product into two telescoping products.
7. Complete the difference of squares factorization in the denominator. Then either manipulate the numerator or guess the form of the telescope.
8. (1978 B2) Fix n and sum over all m by partial fraction decomposition.

*Problems with no source are original, classics, or problems I have forgotten the source of.

4 Trigonometry

Sum Identities For all $\alpha, \beta \in \mathbb{R}$,

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

These two identities can be used to derive a slew of other trig identities.

1. Prove the sine and cosine difference identities and double angle identities:
   (a) $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$
   (b) $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
   (c) $\sin(2x) = 2\sin x\cos x$
   (d) $\cos(2x) = \cos^2 x - \sin^2 x$
2. Prove the tangent sum identity:
$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}.$$
3. Prove the sine and cosine power reduction formulas and product-to-sum formulas:
   (a) $\sin^2 x = (1-\cos(2x))/2$
   (b) $\cos^2 x = (1+\cos(2x))/2$
   (c) $\sin(\alpha)\cos(\beta) = (\sin(\alpha+\beta)+\sin(\alpha-\beta))/2$
   (d) $\sin(\alpha)\sin(\beta) = (\cos(\alpha-\beta)-\cos(\alpha+\beta))/2$
   (e) $\cos(\alpha)\cos(\beta) = (\cos(\alpha-\beta)+\cos(\alpha+\beta))/2$
4. If $\sin(2x) = 24/25$, find the value of $\sin^4 x + \cos^4 x$.
5. A sequence $\{a_n\}$ satisfies $a_1 = 1$, $a_2 = 1/\sqrt{3}$, and for $n \geq 1$, $a_{n+2} = (a_n + a_{n+1})/(1-a_n a_{n+1})$. Find $|a_{2024}|$.
6. Let $a, b \in \mathbb{R}$ such that $0 < ab < 1$. Prove that
$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right).$$
7. Prove that
$$\sum_{k=1}^{n} \arctan\frac{1}{2k^2} = \arctan\frac{n}{n+1}.$$
8. Evaluate
$$\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)dx.$$

Euler's Formula For all $\theta \in \mathbb{R}$,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Furthermore,

$$\cos(\theta) = \mathrm{Re}(e^{i\theta}),\ \sin(\theta) = \mathrm{Im}(e^{i\theta}).$$

1. Prove the sine and cosine triple angle identities:
   (a) $\sin(3x) = 3\sin x - 4\sin^3 x$
   (b) $\cos(3x) = 4\cos^3 x - 3\cos x$
2. Prove the exponential forms of sine and cosine:
   (a) $\sin x = (e^{ix} - e^{-ix})/(2i)$
   (b) $\cos x = (e^{ix} + e^{-ix})/2$
3. Let a and b be distinct integers. Evaluate
$$\int_0^{2\pi} \sin(ax)\cos(bx)\,dx.$$
4. Let n be a positive integer. Prove that
$$\cos\left(\frac{\pi}{2n+1}\right) + \cos\left(\frac{3\pi}{2n+1}\right) + \cdots + \cos\left(\frac{(2n-1)\pi}{2n+1}\right) = \frac{1}{2}.$$
5. Prove that $\alpha = 2\cos(\pi/7)$ satisfies $\alpha^3 + \alpha^2 - 2\alpha - 1 = 0$. Use this to show that $\cos(\pi/7)$ is irrational.
6. Let
$$I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx.$$
   For which integers m, $1 \le m \le 10$, is $I_m \neq 0$?

Sum Identities hints:

1. (a/b) Note that $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

   (c/d) Let $\alpha = \beta = x$.

2. Write $\tan(\alpha+\beta) = \sin(\alpha+\beta)/\cos(\alpha+\beta)$.

3. (a/b) Use cosine double angle identity and $\sin^2 x + \cos^2 x = 1$.

   (c/d/e) Expand the right hand side using sum/difference identities.

4. Note that $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$.

5. What identity looks similar to the recursion relation? Make a substitution to take advantage of that.

6. Let $\alpha = \arctan(a)$, $\beta = \arctan(b)$ in the tangent sum identity.

7. Induct on n.

8. Let $\alpha = x$, $\beta = 1-x$. Our integrand becomes $\arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$.

Euler's Formula hints:

1. (a/b) Note that $e^{i(3\theta)} = (e^{i\theta})^3$. Expand each using Euler's Formula and look at the real and imaginary parts of each.

2. (a/b) Expand the right hand side using Euler's Formula.

3. Rewrite using exponential forms of sine and cosine.

4. Use the fact that $\cos(\theta) = \mathrm{Re}(e^{i\theta})$.

5. Rewrite α using the exponential form of cosine, then expand α^2 and α^3. Prove irrationality by rational root theorem.

6. (1985 A5) Rewrite using exponential form of cosine.

*Problems with no source are original, classics, or problems I have forgotten the source of.

5 Calculus

Intermediate and Mean Value Theorems Let f be continuous on $[a,b]$. Then for all L between $f(a)$ and $f(b)$, there exists an $\alpha \in [a,b]$ such that $f(\alpha) = L$.

Furthermore, if f is differentiable on (a,b), then there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}, \quad f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

The two forms above are equivalent (the derivative of the right gives the left).

1. Let f be a continuous function on $[a,b]$ such that $\int_a^b f(x)dx = 0$. Prove that f has at least one real root.
2. Let f be differentiable on $[0,1]$ and let $f(\alpha) = 0$ and $f(x_0) = -.0001$ for some α and $x_0 \in (0,1)$. Also let $|f'(x)| \geq 2$ on $[0,1]$. Find the smallest upper bound on $|\alpha - x_0|$ for all such functions.
3. If $a_0, a_1, \ldots, a_n$ are real numbers satisfying
$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$
show that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ has at least one real root.
4. Let f be differentiable on $[a,b]$ such that $f(a) = b$ and $f(b) = a$. Prove that there exist two distinct $c, d \in (a,b)$ such that $f'(c)f'(d) = 1$.
5. How many real roots does the function $f(x) = 2^x - 1 - x^2$ have?

Integrand Symmetry Many integrals can be solved by utilizing some symmetry of the integrand, often by a substitution resulting in the integral being expressed in terms of itself.

1. Evaluate
$$\int_0^1 \frac{x^2}{x + \sqrt{1-x^2}}dx.$$
2. Evaluate
$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left(\frac{\pi}{2n}(x_1 + x_2 + \cdots + x_n)\right) dx_1 dx_2 \ldots dx_n.$$
3. Evaluate
$$\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}dx.$$

4. Evaluate
$$\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}\,dx.$$
5. Evaluate
$$\int_0^1 \left((e-1)\sqrt{\ln(1+ex-x)}+e^{x^2}\right)dx.$$
6. Let $f(x) = (x^2-x)/(x^2-3x+1)$. Show that
$$\int_{-100}^{-10} f(x)^2\,dx + \int_{\frac{1}{101}}^{\frac{1}{11}} f(x)^2\,dx + \int_{\frac{11}{10}}^{\frac{101}{100}} f(x)^2\,dx$$
is a rational number.

Feynman's Trick Let $f(x,t)$ and $\frac{\partial f}{\partial t}$ be continuous with respect to x and t. Then
$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx.$$
This can be used to solve tricky single-variable integrals by introducing a second variable t such that cancellation with occur when taking the derivative under the integral.

1. Evaluate
$$\int_0^\infty \frac{\arctan(\pi x)-\arctan x}{x}\,dx.$$
2. Evaluate
$$\int_0^1 \frac{x^2-1}{\ln x}\,dx.$$
3. Evaluate
$$\int_0^{\pi/2} \frac{\arctan(\sin x)}{\sin x}\,dx.$$
4. Evaluate
$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$$

Intermediate and Mean Value Theorems hints:

1. Use the second form of the MVT.
2. (1988 VTRMC) Use the first form of the MVT.
3. (1958 A1) Let f be the given expression. Find the integral of f from 0 to 1.
4. Use MVT on $g(x) = f(f(x))$.
5. (1973 A4) Find easy zeros, then show $f'(x)$ has only two zeros.

Integrand Symmetry hints:

1. (2019 VTRMC) Substitute $x = \cos(u)$ and $x = \sin(u)$.
2. (1965 B1) Substitute $x_i = \pi/2 - u_i$ for $i = 1, \ldots, n$.
3. (1980 A3) Substitute $x = \pi/2 - u$.
4. (2011 VTRMC) Split into two integrals, one from 1 to 2 and another from 2 to 4. Substitute $u = 4/x$ into the second.
5. (2005 VTRMC) Split the sum in the integrand into two integrals. For the first, substitute $u = 1 + ex - x$, graph the result $y = f(u)$, then integrate in terms of y.
6. (1993 A5) Substitute $u = 1/(1-x)$. Guess the form of the antiderivative.

Feynman's Trick hints:

1. (1982 A3) Introduce t so that $I(\pi) = 0$ and $I(1)$ is the desired integral.
2. What derivative will cancel the $\ln x$?
3. Cancel the $\sin x$ in the denominator.
4. (2005 A5) Introduce t inside the ln so that $I(0) = 0$, $I(1)$ is the desired integral, and the result can be integrated using partial fraction decomposition.

*Problems with no source are original, classics, or problems I have forgotten the source of.

6 Number Theory

Modular Arithmetic Let n be a positive integer. For $a, b \in \mathbb{Z}$, we say $a \equiv b \pmod{n}$ if $a - b$ is divisible by n (or, equivalently, if a and b have the same remainder when divided by n).

If $a_1 \equiv a_2 \pmod{n}$ and $b_1 \equiv b_2 \pmod{n}$, then $a_1 + b_1 \equiv a_2 + b_2 \pmod{n}$ and $a_1 \cdot b_1 \equiv a_2 \cdot b_2 \pmod{n}$.

1. Let $S(n)$ be the sum of the digits of a positive integer n. Prove that $S(n) \equiv n \pmod{9}$.
2. Find all integer solutions to $a^2 - 3b^2 = 8$.
3. Let $p(x)$ be a polynomial with integer coefficients such that $p(0)$ and $p(1)$ are odd. Prove that $p(x)$ has no integer roots.
4. Let a, b, n be positive integers. Prove that $a - b$ divides $a^n - b^n$ for all n and $a + b$ divides $a^n + b^n$ if n is odd.
5. Let $S(n)$ be the sum of the digits of a positive integer n. Find $S(S(S(4444^{4444})))$.
6. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

Euler's Totient Theorem Let $\phi(n)$ be Euler's Totient Function (the number of positive integers less than n that are relatively prime to n). If $\gcd(a, n) = 1$, then $a^{\phi(n)} \equiv 1 \pmod{n}$.

To calculate $\phi(n)$, either check manually or use the following formula: if $p_1, \ldots, p_k$ are all of the distinct primes dividing n, then $\phi(n) = n(1 - 1/p_1) \cdots (1 - 1/p_k)$.

1. Calculate $\phi(53), \phi(100)$, and $\phi(144)$.
2. Find the last two digits of $7^{2041} - 3^{2041}$.
3. 2^{29} is a 9-digit integer with distinct digits. Find the digit that is not in the expansion.
4. Find the remainder when $123^{45^{67}}$ is divided by 17.
5. Define $f(1) = 3$ and $f(n+1) = 3^{f(n)}$ for $n \geq 1$. Find the units digit of $f(2024)$.
6. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

Chinese Remainder Theorem Let n and m be coprime positive integers. Then the system $x \equiv a \pmod{n}$, $x \equiv b \pmod{m}$ has a unique solution $x \equiv c \pmod{nm}$. In other words, x modulo n and x modulo m uniquely determine x modulo $n \cdot m$.

1. Find the last two (right-most) digits of 2^{2024}.
2. There exists a positive integer n such that

$$n^{13} = 2581452668046920778582615126663.$$

   Find n.
3. Find the last two (right-most) non-zero digits of 2024!.
4. Do there exist 1,000,000 consecutive integers each of which contains a repeated prime factor?
5. For each positive integer n, let $k(n)$ be the number of 1s in the binary representation of $2023 \cdot n$. What is the minimum value of $k(n)$?
6. Find the least positive integer n for which $2^n + 5^n - n$ is a multiple of 1000.

Modular Arithmetic hints:

1. Let $n = \overline{d_k d_{k-1} \cdots d_0}$. Then $n = \sum_{i=0}^{k} d_i 10^i$.

2. Consider the equation modulo 3.

3. (*1952 A1*) Let $p(x) = \sum_{k=0}^{n} a_k x^k$. What information does $p(0)$ and $p(1)$ give modulo 2?

4. If $a - b = c$, then $a = b + c$. Expand $a^n = (b+c)^n$ modulo c.

5. (1975 IMO) Let $N = 4444^{4444}$. Bound $S(N)$ based on the number of digits of N. Similarly bound $S(S(N))$ and $S(S(S(N)))$.

6. (1966 B2) Prove that if n is not a multiple of 2, 3, 5, or 7, then n is relatively prime to each of $n+1$, $n+2$, $\ldots$, $n+9$.

Euler's Totient Theorem hints:

1. Use the formula.

2. Find $\phi(100)$ to reduce the powers.

3. Find the sum of the digits of 2^{29}.

4. Find 67 modulo $\phi(\phi(17))$.

5. Find $f(2022)$ modulo $\phi(\phi(10))$.

6. (1985 A4) If 3 does not divide n, then $3^a \pmod{n}$ is determined by $a \pmod{\phi(n)}$.

Chinese Remainder Theorem hints:

1. Find 2^{2024} modulo 4 and modulo 25.

2. (2008 Iran) Find rough bounds on n. Consider n^{13} modulo 9 and 11.

3. How many trailing zeroes does 2024! have? After dividing out those zeroes, evaluate modulo 4 and modulo 25.

4. (1955 B4) Set up a system of modular congruences.

5. (2023 B2) Find $a, b \in \mathbb{Z}$ such that $1 + 2^a + 2^b \equiv 0 \pmod{2023}$.

6. (2022 AIME II) For $n \geq 3$, you need $5^n \equiv n \pmod{8}$ and $2^n \equiv n \pmod{125}$.

*Italicized problem sources are slightly modified or served as inspiration. Problems with no source are original, classics, or problems I have forgotten the source of.

7 Recursion

Linear Recursion Suppose $\{a_n\}$ is a sequence with recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $c_i \in \mathbb{R}$. The characteristic polynomial for the sequence is defined by $p(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - \cdots - c_k$. If the roots of $p(x)$, denoted $r_1, \ldots, r_k$, are distinct, then the closed form for the sequence is

$$a_n = d_1 r_1^n + \cdots + d_k r_k^n$$

where $d_1, \ldots, d_k$ are determined by the values of $a_1, \ldots, a_k$.

1. Let $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ $(n \geq 2)$ be the Fibonacci sequence. Prove
$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$
for all $n \geq 0$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio.
2. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive integers such that $a_{n+1} + a_{n-1} = \frac{5}{2} a_n$ for all $n \geq 1$. Find the smallest possible value of a_{100}.
3. Let $a_0 = a_1 = 1$, $a_2 = 2$, and $a_n = 3a_{n-1} - 4a_{n-2} + 2a_{n-3}$ for $n \geq 3$. Find all positive integers k such that $a_k = 2$.
4. Define $x_0 = 0$, $x_1 = 1$, and $x_n = 5x_{n-1} - 6x_{n-2}$ for $n \geq 2$. Prove that $x_n \mid x_{kn}$ for all nonnegative integers n and k.
5. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

Uniqueness Suppose $\{a_n\}, \{b_n\}$ are both sequences recursively defined by (not necessarily equivalent) relations and a_n is determined by $a_0, \ldots, a_k$. If $b_n = a_n$ for all $0 \leq n \leq k$ and b_n satisfies the recursion for a_n, then $b_n = a_n$ for all $n \geq 0$.

This is useful when you are given a recursive sequence and want to show it is equal to another (less complicated) recursive sequence.

1. Let $u_1 = 1/2$ and $u_{n+1} = u_n(n+1)/(2n)$ for $n \geq 1$. Find a closed form for u_n.
2. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,
$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$
The first few terms are $2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392$. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

4. Let $a_0 = 0$, $a_1 = 1$, and $a_{n-1}a_{n+1} - a_n^2 = (-1)^n$. Prove that $a_n = F_n$, the n^{th} Fibonacci number, for all $n \geq 0$.

5. Let $Q_0(x) = 1$, $Q_1(x) = x$, and
$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$
for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

6. Define the sequence $\{a_n\}$ such that, for all $n \geq 1$, a_n is the smallest positive integer satisfying $a_n - \lfloor\sqrt{a_n}\rfloor = n$. Find a closed form for a_n.

Miscellaneous These are miscellaneous recursion problems that can be solved using induction, infinite series, and number theory.

1. Let $R_1 = 1$, and $R_{n+1} = 1 + n/R_n$ for $n \geq 1$. Prove that $0 \leq R_n - \sqrt{n} \leq 1$ for all $n \geq 1$.

2. Let $a_n = x^n + x^{-n}$ for some fixed x such that a_1 is an integer. Prove that a_n is an integer for all $n \geq 1$.

3. Let $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}$ be the Fibonacci sequence. Prove that, for any positive integers m and n, $F_mF_n + F_{m+1}F_{n+1} = F_{m+n+1}$.

4. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in the simplest form.)

5. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

Linear Recursion hints:

1. The characteristic polynomial is $x^2 - x - 1$.
2. All terms are integers, so one term in the closed form must be zero.
3. $(1 \pm i) = \sqrt{2}e^{\pm i\pi/4}$ and $e^{ix} + e^{-ix} = 2\cos x$.
4. If a, b are integers, then $a^{kn} - b^{kn} \equiv a^{kn} - (a^n)^k \equiv 0 \pmod{a^n - b^n}$ for all positive integers n, k.
5. (2015 A2) If k, n are positive integers and k is odd, then $(a^n + b^n) \mid (a^{kn} + b^{kn})$ for all integers a, b.

Uniqueness hints:

1. How does the numerator and denominator of u_{n+1} relate to those of u_n?
2. (1990 A1) Integer powers and factorials are well-known sequences.
3. (2004 A3) Relate u_{n+1} and u_{n-1}.
4. Prove $F_{n-1}F_{n+2} - F_nF_{n+1} = (-1)^n$ by induction.
5. (2017 A2) Write $Q_n(x)$ as a linear combination of $Q_{n-1}(x)$ and $Q_{n-2}(x)$.
6. Write $a_n = x_n^2 + y_n$ for integer sequences x_n, y_n with $0 \le y_n \le 2x_n$. Bound x_n.

Miscellaneous hints:

1. (1958 A2) Use induction.
2. Find a recurrence relation for a_n involving $a_1 a_{n-1}$, then use strong induction.
3. Fix $m = 0$ and prove by induction. Then prove by strong induction for $m > 0$.
4. (1980 B3) Let $u_n = 2^n(a - b_n)$ for some sequence b_n. Find a relation for b_n and its limit.
5. (1998 A4) Define D_n to be the number of digits in A_n. Find a recurrence relation for A_{n+1} involving A_n, A_{n-1}, and D_n.

*Problems with no source are original, classics, or problems I have forgotten the source of.

8 Combinatorics

Recursive Counting We can often count the number of arrangements of n objects in terms of the number of lower-number arrangements.

1. Find the number of sequences of 1's and 3's (taking order into account) that sum to 16.

2. Define a "good word" as a sequence of letters that consists only of the letters A, B, and C (some of these letters may not appear in the sequence) such that A is never immediately followed by a B, B is never immediately followed by a C, and C is never immediately followed by a A. Find the number of ten-letter "good words."

3. Let $a(n)$ be the number of representations of the positive integer n as the sums of 1's and 2's taking order into account. For example, since

$$4 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2,$$

then $a(4) = 5$. Let $b(n)$ be the number of representations of n as the sum of integers greater than 1, again taking order into account and counting the summand n. For example, since

$$6 = 4+2 = 2+4 = 3+3 = 2+2+2,$$

then $b(6) = 5$. Show that for each n, $a(n) = b(n+2)$.

4. In how many ways can the integers from 1 to n, inclusive, be ordered so that (except for the first integer on the left) every integer differs by 1 from some integer to the left of it?

5. Find the number of orderings $(a_1, a_2, \dots, a_9)$ of $(1, 2, \dots, 9)$ such that $a_k \geq a_{k-1} - 2$ for all $k = 2, \dots, 9$.

Bijections If A and B are sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. A bijection is a mapping so that each element of A corresponds to a unique element in B, and vise versa.

1. Suppose you are at the origin of the xy-plane. You are allowed to move only to the right one unit or up one unit. How many different paths can you take to the point $(3,3)$? What about to an arbitrary point (m, n)? (This is a classic combinatoric technique called block walking)

2. A fair 6-sided die is rolled ten times. In how many ways can the die be rolled so that each of the final nine rolls is at least as large as the roll preceding it?

3. Given n objects arranged in a row, a subset of these objects is called "unfriendly" if no two of its elements are consecutive. Show that the number of "unfriendly" subsets each having k elements is

$$\binom{n-k+1}{k}.$$

4. An n-term sequence in which each term in either 0 or 1 is called a binary sequence of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to $0, 1, 0$ in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to $0, 0, 1, 1$ or $1, 1, 0, 0$ in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

Inclusion-Exclusion Principle

Let A and B be two sets and $|S|$ denote the cardinality (number of elements) in a set S. Then

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

or in other words, the number in elements in either A or B is the sum of the elements in A and B minus the elements in both A and B.

1. Find the number of positive integers less than 1001 that are multiples of 2 or 3.

2. Let A, B, C be sets. Prove that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

3. Let n be a positive integer. Find the number of subsets of $\{1, 2, \ldots, 3n\}$ that are subsets of neither $\{1, 2, \ldots, 2n\}$ nor $\{n+1, n+2, \ldots, 3n\}$.

4. A random number selector can only select one of the nine integers $1, 2, \ldots, 9$, and it makes these selections with equal probability. Determine the probability that after n selections ($n > 1$), the product of the n numbers selected will be divisible by 10.

5. A tournament consists of $n \geq 3$ teams. Each team plays one game with every other team, resulting in one winner and one loser with equal probability (there are no ties). Let $p(n)$ denote the probability that some team will end the tournament either win-less or undefeated. Evaluate

$$\lim_{n \to \infty} \frac{2^n \cdot p(n)}{n}.$$

Recursive Counting hints:

1. (2008 VTRMC) Every such sequence must either end with a 1 or with a 3. What must each case sum to without this last integer?

2. (*2003 AIME II*) Let A_n, B_n, C_n be the number of "good words" of length n that end in an A, B, C respectively. What letters may preceed the final A, B, and C respectively?

3. (1957 B4) Find a closed form for $a(n)$ and $b(n)$ by finding a recursive definition. For $a(n)$, a sum will end in a 1 or 2. For $b(n)$, a sum will end in a 2, 3,..., n.

4. (1965 A5) What are all the possibilities for the last integer?

5. (*2006 AIME I*) What are all the possibilities for the integer following n?

Bijections hints:

1. Represent a move upward and right-ward with a U and R respectively. How many orderings of U's and R's are there?

2. (*2001 AIME I*) Represent the problem as a block walk.

3. (1956 A5) For any "unfriendly" sequence $a_1, a_2, \ldots, a_k$, consider subtracting $i-1$ from a_i.

4. (1996 USAMO) Consider summing consecutive terms in b_n.

Inclusion-Exclusion Principle hints:

1. Let A and B be sets of positive integers divisible by 2 and 3 respectively. What integers are in $A \cap B$?

2. Treat $B \cup C$ as the second set in the definition.

3. (*2017 AIME II*) Use complementary counting. How many subsets are there of a set of size k? If a set S is a subset of both $\{1, 2, \ldots, 2n\}$ and $\{n+1, n+2, \ldots, 3n\}$, what other set must it be a subset of?

4. (1972 USAMO) Find the probability that the probability won't be divisible by 10.

5. (*1996 AIME*) How many teams can go win-less? How many teams can go undefeated?

*Italicized problem sources are slightly modified or served as inspiration. Problems with no source are original, classics, or problems I have forgotten the source of.

9 Probability

Geometric Probability The probability that a point chosen uniformly at random from a region Y is in a subregion X is $V(X)/V(Y)$, where V denotes the appropriate measurement (length for 1-dimension, area for 2-dimension, volume for 3-dimension).

1. The unit circle is inscribed in a square with side length 2. Find the probability that a random point chosen within the square lies outside the circle.
2. A coin of diameter $1/2$ is tossed randomly onto an infinite unit grid (a grid of squares of side length 1). Find the probability that the coin is not touching any of the grid lines.
3. Let a and b be chosen independently and uniformly at random from the interval $(0,1)$. Find the probability that $\lfloor \log_2 a \rfloor = \lfloor \log_2 b \rfloor$. ($\lfloor x \rfloor$ denotes the largest integer not exceeding x)
4. Find the probability that the sum of two randomly chosen numbers in the interval $[0,1]$ does not exceed 1 and their product does not exceed $2/9$.
5. Let a and b be given positive real numbers with $a < b$. If two points are selected at random from a straight line segment of length b, what is the probability that the distance between them is at least a?
6. Two real numbers x and y are chosen at random in the interval $(0,1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

Probability States Let $S_1, S_2, \ldots, S_n$ be the n possible states of some process with S_n being the "goal" state. Let $P_1, P_2, \ldots, P_{n-1}$ be the probability of reaching S_n from state $S_1, S_2, \ldots, S_{n-1}$ respectively. We can express P_i in terms of previous (or adjacent) states, then solve the resulting system.

1. A game is played on a number line. Starting at 1, you flip a (fair) coin. If it is heads, you move one unit to the right. If it is tails, you move one unit to the left. You lose the game if you move to 0 and you win the game if you move to 4. Find the probability that you will win the game.
2. A bug starts at a vertex of a triangle. On each move, it randomly selects one of the two vertices where it is not currently located and crawls along the side of the triangle to that vertex. Find the probability that the bug moves to its starting vertex on its tenth move.

3. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

4. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

5. A player throwing a 6-sided die scores as many points as on the top face of the die and is to play until their score reaches or passes a total n. Denote by $p(n)$ the probability of making exactly the total n. Find the value of $\lim_{n\to\infty} p(n)$.

Expected Value States Let $S_1, S_2, \ldots, S_n$ be the n possible states of some process with S_n being the "goal" state. Let $E_1, E_2, \ldots, E_{n-1}$ be the expected number of "moves" to reach state S_n from state $S_1, S_2, \ldots, S_{n-1}$ respectively. We can express E_i in terms of adjacent states, then solve the resulting system.

1. Find the expected number of (fair) coin flips to get two heads in a row.

2. A particle starts in the center of a 3×3 grid. Every second it moves with equal probability to one of the four squares adjacent to its current position: directly up, down, left, or right. Find the expected number of seconds until the particle exits the grid.

3. Alice and Bob play a game starting with no coins each. A player wins a round by gaining coins. On round n, Alice and Bob randomly choose an integer from 0 to n, inclusive. If they both choose the same integer, the player that won the previous round receives $n+2$ coins. Otherwise, the player that lost the previous round receives 1 coin. On round 1, Alice is considered to have "won" the previous round. Find the expected number of coins Alice will have after 10 rounds.

4. Let $A_1A_2\ldots A_{12}$ be a dodecagon (12-gon). Three frogs initially sit at A_4, A_8, and A_{12}. At the end of each minute, simultaneously, each of the three frogs jumps to one of the two vertices adjacent to its current position, chosen randomly and independently with both choices being equally likely. All three frogs stop jumping as soon as two frogs arrive at the same vertex at the same time. Find the expected number of minutes until the frogs stop jumping.

Geometric Probability hints:

1. Find the probability the point lies within the circle, then subtract that from 1.

2. (*Putnam and Beyond*) Where must the center of the coin land for it to not touch any grid lines?

3. (2017 AMC 12B) For any fixed integer k, find the probability that $\lfloor \log_2 x \rfloor = k = \lfloor \log_2 y \rfloor$.

4. (Putnam and Beyond) Graph the inequalities on the xy-plane for $x, y \in [0,1]$.

5. (1961 B2) It is equivalent to find, for randomly selected x and y with $0 \le y < x \le b$, the probability that $x - y \ge a$.

6. (1993 B3) Suppose $2k$ is the closest integer. Bound y in terms of x and k, then graph.

Probability States hints:

1. For $n = 1,2,3$, let $P(n)$ be the probability of winning the game if you start from point n. At 1, you lose the game with probability $1/2$ and move to 2 with probability $1/2$. Thus, $P(1) = (1/2)(0) + (1/2)P(2)$. Express $P(2)$ and $P(3)$ similarly, then solve for $P(1)$.

2. (2003 AIME II) Let $P(n)$ be the probability that the bug moves to its starting vertex on its n^{th} move. On move $n-1$, the bug can be either on its starting vertex (with probability $P(n-1)$) or on one of the other two vertices (with probability $1-P(n-1)$). What is the probability of moving to the starting vertex from each of these positions? Find $P(1)$.

3. (2001 A2) Let $P(n)$ be the probability of flipping an odd number of heads when tossing coins $C_1, \ldots, C_n$. Express $P(n)$ in terms of $P(n-1)$. Find $P(1), P(2), P(3), \ldots$, and prove the closed form using the recurrence.

4. (2002 B1) Let $P(k,n)$ be the probability that Shanille hits exactly k of her first n shots. Express $P(k,n)$ in terms of $P(k-1,n-1)$ and $P(k,n-1)$. Find $P(1,3)$, $P(2,3)$, $P(1,4)$, $P(2,4)$, $\ldots$, and prove the closed form for $P(k,n)$ using the recurrence.

5. (1960 A6) Define $p(0) = 1$ and $p(n) = 0$ for $n < 0$. We can express $p(n)$ in terms of $p(n-1), \ldots, p(n-6)$. Adding all of these recurrences for $n = 0, 1, \ldots, n$ yields a recurrence for $p(n)$ in terms of $p(n-1), \ldots, p(n-5)$. Prove $p(n)$ converges to finish.

Expected Value States hints:

1. Let S_1 denote when the last flip was tails, S_2 when the last flips were tails then heads, and S_3 when the last flips were both heads. Suppose we are in state S_1. Then $E_1 = (1/2)(E_1+1)+(1/2)(E_2+1)$ since we can either flip tails (with probability $1/2$) and remain in S_1, or we can flip heads (with probability $1/2$) and move to S_2. Express E_2 similarly and solve for E_1.

2. (*2017 AMC 12A*) Let O, E, C denote the expected number of seconds to exit the grid when at the origin, an "edge" (one move from origin), and a "corner" (one diagonal move from origin), respectively.

3. The game is fair, so the expected number of coins that Alice has is equal to the expected number of coins that Bob has. Find the expected number of coins between the both of them.

4. (2021 AIME I) Let $E(a, b, c)$ denote the expected number of minutes until the frogs stop jumping when the pairwise distances between the frogs are $a \geq b \geq c$. How many states are possible?

*Italicized problem sources are slightly modified or served as inspiration. Problems with no source are original, classics, or problems I have forgotten the source of.

10 Geometry

2-D Geometry These are problems concerning geometry in two dimensions.

1. Show that if three points are inside a closed unit square, then some pair of them are within $\sqrt{6}-\sqrt{2}$ units apart.
2. Two squares of side length 0.9 are placed inside a circle of radius 1. Prove that the squares overlap.
3. Given a set of 6 points in the plane, prove that the ratio of the longest distance between any pair to the shortest is at least $\sqrt{3}$.
4. Given five points in a plane, no three of which lie on a straight line, show that some four of these points form the vertices of a convex quadrilateral.
5. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

3-D Geometry These are problems concerning geometry in three dimensions.

1. Find the radius of the smallest sphere that can contain a rectangular box with side lengths 3, 4, and 5.
2. Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.
3. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A sphere of radius 2 rests on them. What is the distance from the plane to the top of the larger sphere?
4. What is the average squared straight line distance between two points on a sphere of radius 1?
5. What is the average straight line distance between two points on a sphere of radius 1?

Vectors Vectors can be used to represent distances in both 2-D and 3-D problems. An important tool to use with vectors is the dot product:

$$\vec{a}\cdot\vec{b} = a_1b_1 + \cdots + a_nb_n = |\vec{a}||\vec{b}|\cos\theta$$

where θ is the smaller angle between $\vec{a}$ and $\vec{b}$.

1. In equilateral ΔABC let points D and E trisect $\overline{BC}$. Find $\sin(\angle DAE)$.
2. Given an arbitrary triangle ABC, construct equilateral triangles ABZ, ACY, and BCX externally on the sides of the triangle ABC. Prove that $AX = BY = CZ$.
3. The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.
4. Given n points on the sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$, prove that the sum of the squares of the distances between them does not exceed n^2.

2-D Geometry hints:

1. (1960 A2) Without loss of generality, we can let one point be the corner of a square of side length $s \leq 1$.

2. (Putnam and Beyond) Prove that for any two (identical) squares inside of a circle, one must be contained in half of the circle.

3. (1964 A1) Prove that there exists a triangle with vertices in the 6 points with one internal angle greater than $120°$. Then use law of cosines on this triangle.

4. (1962 A1) A quadrilateral is convex if and only if its inner diagonals intersect.

5. (1966 B1) The convex condition ensures that any vertical (or horizontal) line within the square intersects P at most twice.

3-D Geometry hints:

1. The radius of the sphere will be half the inner diagonal of the box.

2. (1983 B1) Find R without the restriction of being between S and C.

3. (2004 AMC 12A) Use pythagorean theorem to find the distance between the center of the larger sphere and the plane containing the centers of all the smaller spheres.

4. (*1958 B4*) Without loss of generality let one point be the bottom of the sphere. How do the squared distances of a point on the bottom half of the sphere relate to the reflection of the point to the upper half?

5. (1958 B4) Without loss of generality let one point be the bottom of the sphere.

Vectors hints:

1. (2013 AIME II) Let A be at the origin and C on the x-axis. Find the coordinates of D and E, then write $\overline{AD}$ and $\overline{AE}$ as vectors.

2. Let R be the matrix transformation that rotates $60°$ clockwise. Write $\overrightarrow{AX} + R \cdot \overrightarrow{BY}$ in terms of $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$.

3. (1983 A2) Express each hand as a vector, and think of the long hand as fixed with the short hand rotating. Then the velocity vector of the short hand has constant magnitude. How does this vector relate to the rate of change of the hands' distance?

4. (1968 A4) Express the distance between two points as an expression involving the dot product of two vectors, then bound the resulting sum using the trivial inequality for vectors.

*Italicized problem sources are slightly modified or served as inspiration. Problems with no source are original, classics, or problems I have forgotten the source of.